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Fibrations of compact Riemannian manifolds

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Introduction

In [10] A. LICHNEROWICZ proved that every compact oriented Riemannian manifold M with nonnegative generalized Ricci tensor is the total space of a fibre bundle with flat torus base space and with totally geodesic bundle projection. He also showed that the universal covering $\tilde{M}$ of M splits isometrically as $\mathbf{R}^k \times M_0$ where M_0 is compact and satisfies the same curvature condition as M and $\mathbf{R}^k$ is endowed with the flat metric. In [1] J. CHEEGER and D. GROMOLL proved that there is a finite covering $\hat{M}$ of M such that $\hat{M}$ is diffeomorphic to the product of a flat torus $\mathbf{T}^k$ and an another compact manifold M_1. However, in many cases, $\hat{M}$ does not split isometrically as $\mathbf{T}^k \times M_1$.

These results can be obtained by the study of certain harmonic mappings and their relations with the isometry group of M. The subject of our present note is to generalize the theorems mentioned above to the case when there is no curvature assumption of M. In this way we obtain several results about the structure of compact Riemannian manifolds and their covering spaces.

The body of the paper is divided into two parts:

In Part I we overview some properties of harmonic mappings and their relations with the isometry group. This part is essentially based on [10]. In Part II we study compact Riemannian manifolds in general and then we apply the obtained results to compact homogeneous Riemannian manifolds and compact Lie groups. All manifolds, mappings, bundles, etc. are supposed to be smooth, i.e. of class C^∞, unless stated otherwise.

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I. Harmonic mappings and their factorizations

1. The notion of harmonic mappings. Let $W \to M$ be a vector bundle over a Riemannian manifold M and denote $\mathscr{T}^{[r]}(M) \to M$, $r \in \mathbf{N}$, the bundle of r-covectors of M. Further put

$$\Lambda^0(M, W) = \operatorname{Sec} W \quad \text{and} \quad \Lambda^r(M, W) = \operatorname{Sec}(W \otimes \mathscr{T}^{[r]}(M)), \quad r \in \mathbf{N}.$$

The elements of $\Lambda^r(M, W)$ are called r-forms on M with values in W.

A covariant differentiation on the vector bundle $W \to M$ is a linear mapping

$$\nabla \colon \Lambda^0(M, W) \to \Lambda^1(M, W)$$

which satisfies the derivation rule

$$\nabla(\mu w) = w \otimes d\mu + \mu \nabla w,$$

$w \in \Lambda^0(M, W)$ and μ scalar on M. The operator ∇ defines a covariant differentiation

$$i_X \circ \nabla = \nabla_X \colon \Lambda^0(M, W) \to \Lambda^0(M, W)$$

for every vector field $X \in \mathfrak{X}(M)$ on M. It can be canonically extended to a covariant differentiation

$$\nabla_X \colon \Lambda^r(M, W) \to \Lambda^r(M, W), \quad r \in \mathbf{N},$$

by the rule

$$\nabla_X(w \otimes \lambda) = (\nabla_X w) \otimes \lambda + w \otimes (\nabla_X \lambda),$$

$w \in \Lambda^0(M, W)$ and $\lambda \in \Lambda^r(M)$.

Now consider a fixed covariant differentiation ∇ of the vector bundle $W \to M$. The exterior differentiation of the vector bundle $W \to M$ is a linear mapping

$$d \colon \Lambda^r(M, W) \to \Lambda^{r+1}(M, W), \quad r = 0 \quad \text{or} \quad r \in \mathbf{N},$$

for which

$$d(w \otimes \lambda) = (\nabla w) \wedge \lambda + w \otimes d\lambda$$

holds, $w \in \Lambda^0(M, W)$ and $\lambda \in \Lambda^r(M)$.

Suppose that the bundle $W \to M$ is Riemannian-connected, i.e. each of the fibres has a positive-definite inner product $(\ ,\ )$ and the covariant differentiation ∇ preserves the metric on the fibres of W, i.e.

$$\nabla_X(w, w') = (\nabla_X w, w') + (w, \nabla_X w')$$

holds, $w, w' \in \Lambda^0(M, W)$ and $X \in \mathfrak{X}(M)$. This inner product can be extended to an inner product of the bundle $W \otimes \mathscr{T}^{[r]}(M) \to M$ by

$$(w \otimes \lambda, w' \otimes \lambda') = (w, w')(\lambda, \lambda'),$$

$w, w' \in \Lambda^0(M, W)$ and $\lambda, \lambda' \in \Lambda^r(M)$.

Let M be compact and oriented and denote its volume element by $v \in \Lambda^n(M)$, $n = \dim M$. The global scalar product of $\Phi, \Psi \in \Lambda^r(M, W)$ is

$$\langle \Phi, \Psi \rangle = \int_M (\Phi, \Psi)\, v.$$

Let

$$\partial\colon \Lambda^r(M,W) \to \Lambda^{r-1}(M,W), \quad r \in \mathbf{N},$$

be the adjoint operator of d with respect to the global scalar product and put $\partial = 0$ if $r = 0$. Finally let

$$\Delta = d \circ \partial + \partial \circ d\colon \Lambda^r(M,W) \to \Lambda^r(M,W), \quad r = 0 \quad \text{or} \quad r \in \mathbf{N},$$

be the Laplace operator of the bundle $W \to M$. An r-form Φ on M with values in W is said to be harmonic if $\Delta\Phi = 0$.

An explicit formula for the operator ∂ is

$$\partial\Phi = -\operatorname{trace}\{(X, Y) \to \nabla_X \circ \imath_Y \Phi\} = -\operatorname{trace}\{(X, Y) \to \imath_X \circ \nabla_Y \Phi\},$$

$\Phi \in \Lambda^r(M, W)$, cf. [14], Proposition (1.1).

Now let M denote a compact and oriented Riemannian manifold and let M' be a complete Riemannian manifold. If $f\colon M \to M'$ is a mapping of class C^2 then let $F \to M$ be the vector bundle obtained by pulling back the tangent bundle $T(M') \to M'$ along f. Then the elements of $\Lambda^0(M, F)$ are canonically identified with the vector fields along f and the tangent map f_* can be considered as a specific 1-form on M with values in F. The covariant differentiation of M' canonically induces a covariant differentiation of the bundle $F \to M$. The metric tensor of M' determines a positive definite inner product on the fibres of $F \to M$ and so the bundle $F \to M$ becomes a Riemannian-connected bundle. The mapping $f\colon M \to M'$ is said to be harmonic if $\Delta f_* = 0$ or equivalently if $\partial f_* = 0$. (Cf. [4] and [10] § 18/c, p. 75.)

2. The ideal I_i and the mapping $\mathscr{J}$ ([10] § 17 and § 19). Let M be a compact oriented Riemannian manifold of dimension n with first Betti number $p = b_1(M)$. The metric tensor of M defines a module isomorphism $\gamma\colon \mathfrak{X}(M) \to \Lambda^1(M)$. Let G_i be the maximal connected subgroup of the group of isometries of M. Then the Lie algebra L_i of G_i can be identified with the Lie algebra of the infinitesimal isometries of M. Then $X \in L_i$ if and only if $\nabla\gamma(X) \in \Lambda^2(M)$. Denote by $\mathscr{H}$ the linear space of harmonic 1-forms of M with dimension p. Every harmonic 1-form is invariant by L_i and hence $\imath_X \alpha$, $X \in L_i$ and $\alpha \in \mathscr{H}$, is a constant function on M. Let $I_i = \{X \in L_i \mid \imath_X \alpha = 0$ for every $\alpha \in \mathscr{H}\}$. Then $[L_i, L_i] \subset I_i$ and $I_i \subset L_i$ is an ideal such that L_i/I_i is commutative. If $X \in L_i$ has a critical point on M then $X \in I_i$.

Consider the universal covering $\pi_M\colon \tilde{M}\to M$ represented by the homotopy classes of curves starting from a base point $m_0\in M$ and denote $\tilde{m}_0\in\tilde{M}$ the class of null-homotopic loops. The mapping π_M pulls back the metric tensor of M to a metric tensor of $\tilde{M}$. Let $\mathscr{H}^*$ be the dual space of $\mathscr{H}$ endowed with the flat metric and define $\tilde{\mathscr{J}}_M\colon \tilde{M}\to\mathscr{H}^*$ by

$$(\tilde{\mathscr{J}}_M(\tilde{m}),\alpha)=u(\tilde{m})-u(\tilde{m}_0),$$

where $\pi_M^*\beta=du$. Let $P\subset\mathscr{H}^*$ be the image of $H^1(M;\mathbf{Z})$ under the de Rham isomorphism $H^1(M;\mathbf{R})\to\mathscr{H}^*$. Then P is a discrete subgroup of $\mathscr{H}^*$ of maximal rank. The canonical torus of M is the quotient $B(M)=\mathscr{H}^*/P$ endowed with the flat metric. The mapping $\tilde{\mathscr{J}}_M$ projects down to a harmonic mapping $\mathscr{J}_M$ such that $p_M\circ\tilde{\mathscr{J}}_M=\mathscr{J}_M\circ\pi_M$, where $p_M\colon \mathscr{H}^*\to B(M)$ is the canonical projection.

3. Factorization of harmonic mappings ([10] § 19). Let M and N be compact oriented Riemannian manifolds with base points $m_0\in M$ and $x_0\in N$, resp. and let $f\colon M\to N$ be a base point preserving map such that $f^*\colon \Lambda^1(N)\to\Lambda^1(M)$ sends harmonic 1-forms of N to harmonic 1-forms of M. Denote the dual map of the restriction of f^* to $\mathscr{H}_N$ by the same symbol $f^*\colon \mathscr{H}_M^*\to\mathscr{H}_N^*$. The base point preserving map f can be lifted to a map $\tilde{f}\colon \tilde{M}\to\tilde{N}$. By the Stokes' theorem $\tilde{\mathscr{J}}_N\circ f=f^*\circ\tilde{\mathscr{J}}_M$ holds. Each face of the cube in Figure 1 commutes, where $f^*\colon \mathscr{H}_M^*\to\mathscr{H}_N^*$ is projected to an affine map $B(f)\colon B(M)\to B(N)$. The bottom face also commutes. Especially, if $M=N$ and $f\colon M\to M$ is an isometry then f induces an affine transformation of $B(M)$.

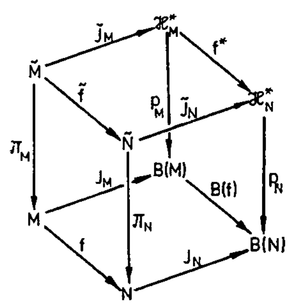

Figure 1

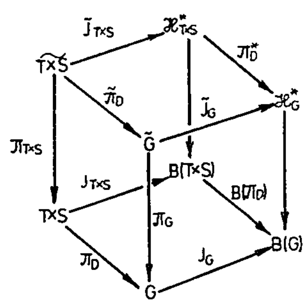

Figure 2

Let $\mathfrak{P}$ be the class of compact Riemannian manifolds such that M belongs to $\mathfrak{P}$ if the quadratic form defined by the symmetric 2-tensor

$$C_{ij}=R_{ij}-\nabla_i\nabla_j\log\lambda$$

is positive semidefinite at every point of M, where λ is some positive scalar on M and R_{ij}'s are the local components of the Ricci tensor. If N belongs to $\mathfrak{P}$ and

$f\colon M\to N$ is a harmonic mapping then f^* sends harmonic 1-forms of N to harmonic 1-forms of M. So, f defines an affine mapping $B(f)\colon B(M)\to B(N)$ making the above diagram commutative. By local calculation it can be shown that if $X\in L_i$ then $\mathscr{I}_*(X)$ defines a uniform vector field on M. Another definition of the ideal I_i can be obtained in this way putting $I_i=\{X\in L_i|\mathscr{I}_*(X)=0\}$. Denote G_B the group of translations of $B(M)$ and let L_B be the Lie algebra of G_B. There is a canonical homomorphism

$$\hat{\mathscr{I}}_i\colon G_i\to G_B$$

satisfying $\mathscr{I}\circ g=\hat{\mathscr{I}}_i(g)\circ\mathscr{I}$, $g\in G_i$. The kernel Γ_i of $\hat{\mathscr{I}}_i$ is a closed invariant subgroup of G_i and the Lie algebra of the maximal connected subgroup $(\Gamma_i)_0$ of Γ_i is I_i. Denote $Z_0\subset G_i$ the maximal connected subgroup of the center of G_i. Then Z_0 is a closed invariant subgroup of G_i and its Lie algebra $\mathfrak{z}_i$ is the center of L_i. The Lie algebra of the closed invariant subgroup $Z_0\cap(\Gamma_i)_0$ is the ideal $\mathfrak{z}_i\cap I_i$. Choose a base $\{Z^1,\dots,Z^r\}$ of $\mathfrak{z}_i\cap I_i$ and let $Z^{r+1},\dots,Z^{r+q}\in\mathfrak{z}_i$ be such that $\{Z^1,\dots,Z^{r+q}\}$ is a base of $\mathfrak{z}_i$. Let $\|\cdot\|$ be a norm on the vector space $\mathfrak{z}_i$. Then there exists $\varepsilon>0$ such that if $X^1,\dots,X^{r+q}\in\mathfrak{z}_i$ with $\|X^j-Z^j\|<\varepsilon$, $j=1,\dots,r+q$, then $\{X^1,\dots,X^{r+q}\}$ is also a base of $\mathfrak{z}_i$. There exist vectors $V^1,\dots,V^q\in\mathfrak{z}_i$ with $\|Z^{r+j}-V^j\|<\varepsilon$ such that $\exp(\mathbf{R}V^j)\subset Z_0$ is closed in Z_0 $(j=1,\dots,q)$, cf. [2], Ch. XIX, Sec. 10, p. 188. Then $\{Z^1,\dots,Z^r,V^1,\dots,V^q\}$ is a base of $\mathfrak{z}_i$. Let $P_i\subset L_i$ be the linear subspace spanned by the vectors V^j, $j=1,\dots,q$. By $P_i\subset\mathfrak{z}_i$, the linear subspace P_i is an ideal of L_i. Furthermore $P_i\cap I_i=\{0\}$ and $P_i+I_i=L_i$ hold. Let $Q_i\subset G_i$ be the connected subgroup which corresponds to the Lie algebra P_i. By $P_i\subset\mathfrak{z}_i$, it follows that $Q_i\subset Z_0$, i.e. Q_i is a central subgroup of G_i. Define $Q_i'=\prod_{j=1}^{q}\exp(\mathbf{R}V^j)$. Then $Q_i'\subset Q_i$ and Q_i' is a closed subgroup of G_i. Because $\dim Q_i=\dim Q_i'=q$ it follows that Q_i' is relatively open in Q_i and hence $Q_i=Q_i'$. Especially we obtain that $Q_i\subset G_i$ is a closed central subgroup. It is easy to see that $G_i=Q_i(\Gamma_i)_0$ holds. The Lie algebra of the closed subgroup $H_i=Q_i\cap\Gamma_i$ is trivial and hence H_i is a finite central subgroup of G_i.

II. Compact oriented Riemannian manifolds as total spaces of fibre bundles

4. The general case. Let M be a compact oriented Riemannian manifold and choose a toroidal subgroup $Q_i\subset G_i$ as in § 3. The subgroup H_i is the kernel of the homomorphism $\hat{\mathscr{I}}=\hat{\mathscr{I}}_i|Q_i\colon Q_i\to G_B$ and if we put $H_B=\operatorname{im}\hat{\mathscr{I}}$ then $\hat{\mathscr{I}}\colon Q_i\to H_B$ is a local isomorphism of compact groups. If $n\in M$ and $y\in B(M)$ then denote $\theta(n)$ and $\vartheta(y)$ the orbit of the action of Q_i and H_B through the point $n\in M$ and $y\in B(M)$, respectively. There is a strong relation between the action of Q_i on M and the action of H_B on $B(M)$ as follows:

Theorem 1. *The mapping* $\mathscr{I}\colon M\to B(M)$ *is equivariant with respect to the epimorphism* $\hat{\mathscr{I}}\colon Q_i\to H_B$. *For every* $m_0\in M$ *the restriction* $\mathscr{I}|\theta(m_0)\colon \theta(m_0)\to\vartheta(\mathscr{I}(m_0))$ *is a finite covering with multiplicity equal to the index of the isotropy subgroup* $(Q_i)_{m_0}\subset H_i$ *in the group* H_i.

Proof. By the equation $\mathscr{I}\circ g=\hat{\mathscr{I}}(g)\circ\mathscr{I}$, $g\in Q_i$, it follows that $\mathscr{I}$ is equivariant with respect to $\hat{\mathscr{I}}$. Let $m_0\in M$ be fixed and show that $\mathscr{I}|\theta(m_0)\colon \theta(m_0)\to\vartheta(y_0)$, $y_0=\mathscr{I}(m_0)$, is a covering. Because $\hat{\mathscr{I}}\colon Q_i\to H_B$ is an epimorphism, $\mathscr{I}|\theta(m_0)\colon \theta(m_0)\to\vartheta(y_0)$ is surjective. Let $y\in\vartheta(y_0)$ be fixed and find a neighbourhood V_B of y in $\vartheta(y_0)$ evenly covered by $\mathscr{I}|\theta(m_0)$, cf. [13], Ch. 2. Sec. 1, p. 62. The subgroup H_i is discrete in Q_i and hence there exists an open and connected neighbourhood U of e in Q_i with $(U\cdot U^{-1})\cap H_i=\{e\}$. Putting $U_B=\hat{\mathscr{I}}(U)$ we obtain a diffeomorphism $\hat{\mathscr{I}}\colon U\to U_B$. The action of H_B on $B(M)$ is free and hence $V_B=U_B(y)$ is an open and connected neighbourhood of y in $\vartheta(y_0)$. Then $(\mathscr{I}|\theta(m_0))^{-1}(V_B)=\cup\{U(m)|m\in(\mathscr{I}|\theta(m_0))^{-1}(y)\}$ and $U(m)\cap U(m')=\emptyset$ if $m\neq m'$, $m,m'\in(\mathscr{I}|\theta(m_0))^{-1}(y)$. We obtain that V_B is evenly covered by $\mathscr{I}|\theta(m_0)$. Thus $\mathscr{I}|\theta(m_0)\colon \theta(m_0)\to\vartheta(y_0)$ is a covering with multiplicity $\operatorname{card}((\mathscr{I}|\theta(m_0))^{-1}(y))=\operatorname{card}(H_i/(Q_i)_{m_0})$ which accomplishes the proof.

Our next result is a generalization of a fibration theorem of A. Lichnerowicz in [10], § 23, pp. 85.

Theorem 2. *Let M be a compact oriented Riemannian manifold and suppose that the rank of the mapping $\mathscr{I}$ is $\leq q=\operatorname{codim} I_i$ at every point of M. Then there exists a harmonic fibration $\bar{\mathscr{I}}\colon M\to\vartheta$ with q-dimensional flat torus base space ϑ and finite commutative structure group. Moreover, $p=q$ holds.*

Proof. By Theorem 1, it follows that $\mathscr{I}$ is a mapping of constant rank q. Thus $\operatorname{im}\mathscr{I}$ consists of a unique orbit of H_B. Because H_B is compact, $\vartheta=\operatorname{im}\mathscr{I}\subset B(M)$ is a flat torus of dimension q and H_B is its group of translations. Let $\iota_\vartheta\colon \vartheta\subset B(M)$ be the canonical embedding and let $\bar{\mathscr{I}}\colon M\to\vartheta$ be defined by $\mathscr{I}=\iota_\vartheta\circ\bar{\mathscr{I}}$. Because $\mathscr{I}$ is harmonic and ι_ϑ is totally geodesic, it follows that $\bar{\mathscr{I}}$ is a harmonic map, [4]. If $y\in\vartheta$ then $\bar{\mathscr{I}}^{-1}(y)$ is a closed submanifold of M. Let $U\subset Q_i$ and $U_B\subset H_B$ be open and connected neighbourhoods of the identity elements, respectively, such that $\hat{\mathscr{I}}|U\colon U\to U_B$ is a diffeomorphism. If $y_0\in\vartheta$ then $V_B=U_B(y_0)$ is an open and connected neighbourhood of y_0 in ϑ. Define a map

$$h\colon \bar{\mathscr{I}}^{-1}(V_B)\to V_B\times\bar{\mathscr{I}}^{-1}(y_0)$$

as follows:

If $m\in\bar{\mathscr{I}}^{-1}(V_B)$ then $\bar{\mathscr{I}}(m)=h(y_0)$ is valid for some $h\in U_B$. There exists $g\in U$ such that $\hat{\mathscr{I}}(g)=h$. Define $h(m)=(\bar{\mathscr{I}}(m), g^{-1}(m))$. Routine calculation shows that h is a bundle map and thus $\bar{\mathscr{J}}\colon M\to\vartheta$ is a locally trivial fibre bundle. It is easy to see that H_i is the structure group of this bundle, cf. [10], Ch. V, § 13, p. 64. By similar reasonings as in [10] we can prove that the fibres of $\bar{\mathscr{J}}\colon M\to\vartheta$ are connected. By the very definition of I_i it follows that $p=q$. Thus the theorem is proved.

Now we turn our attention to the study of covering spaces of compact Riemannian manifolds. In [1] J. Cheeger and D. Gromoll showed that every compact Riemannian manifold of nonnegative Ricci curvature has a finite covering which splits as the product of a torus and of another manifold. Our analogous result is the following:

Theorem 3. *Let M be compact and oriented Riemannian manifold and suppose that the following conditions are satisfied:*

(1) *There is no exceptional orbit of the action Q_i on M.*

(2) *The rank of the mapping $\mathscr{J}$ is $\leqq q=\operatorname{codim} I_i$ at every point of M.*

Then there are finite isometric coverings $\pi\colon M_1\to M$ and $\varrho\colon M\to M_2$ such that M_j $(j=1,2)$ is diffeomorphic with $\mathbf{T}^q\times\overline{M}_j$, where $\overline{M}_j$ is a compact manifold. In diagram:

$$\mathbf{T}^q\times\overline{M}_1\approx M_1\xrightarrow{\pi} M\xrightarrow{\varrho} M_2\approx\mathbf{T}^q\times\overline{M}_2.$$

Proof. By Theorem 1 every orbit of Q_i on M is of highest dimension q. By (1) it follows that every orbit of Q_i on M is principal, i.e. there exists a unique subgroup $K\subset H_i$ such that $(Q_i)_m=K$ holds for every $m\in M$. By virtue of a theorem of [7], Kap. I, § 1.5, p. 6, we obtain a differentiable principal fibre bundle $\mathscr{I}\colon M\to M/Q_i$ with structure group Q_i/K. Because $K\subset H_i$, a routine calculation shows that $\varrho\colon M\to M/H_i$ is a finite covering of multiplicity $\operatorname{card}(H_i/K)$. The space M/H_i can be endowed with a structure of Riemannian manifold so that $\varrho\colon M\to M/H_i$ is a local isometry.

Let $y_0\in\vartheta$ be fixed and consider the fibre $\mathscr{F}_{y_0}=\bar{\mathscr{J}}^{-1}(y_0)$. Then $\mathscr{F}_{y_0}$ is invariant under the action of H_i and thus $\varrho|\mathscr{F}_{y_0}\colon \mathscr{F}_{y_0}\to\mathscr{F}_{y_0}|H_i$ is a finite covering. The inclusions $i\colon\mathscr{F}_{y_0}\subset M$ and $H_i\subset Q_i$ induce a canonical map $\lambda\colon\mathscr{F}_{y_0}|H_i\to M/Q_i$ such that $\mathscr{I}\circ i=\lambda\circ(\varrho|\mathscr{F}_{y_0})$ holds. It is easy to prove that λ is a diffeomorphism. The map $f=(\bar{\mathscr{J}},\mathscr{I})\colon M\to\vartheta\times(M/Q_i)$ is invariant under the action of H_i on M and hence it can be factorized by ϱ yielding a diffeomorphism $\bar{f}\colon M/H_i\to\vartheta\times(M/Q_i)$. Putting $M_2=M/H_i$ and $\overline{M}_2=M/Q_i$ we have

$$M\xrightarrow{\varrho} M_2\approx\mathbf{T}^q\times\overline{M}_2$$

where ϱ is a finite covering.

Let $\pi\colon Q_i\times\mathscr{F}_{y_0}\to M$ be defined by $\pi(g,m)=g(m)$, $g\in Q_i$ and $m\in\mathscr{F}_{y_0}$. Then π is surjective and we show that π is a finite covering. Let $m_0\in M$ be fixed and find a neighbourhood U of m_0 evenly covered by π. Because $\varrho|\mathscr{F}_{y_0}\colon \mathscr{F}_{y_0}\to\mathscr{F}_{y_0}/H_i$ is a finite covering there exists an open and connected neighbourhood S of m_0 in $\mathscr{F}_{y_0}$ such that if $s\in S$ then $H_i(s)\cap S=\{s\}$ holds. (S is a slice of the action of Q_i on M but we shall not use this fact later.) Choose a neighbourhood W of e in Q_i with $(W\cdot W^{-1})\cap H_i=\{e\}$ and put $U=W(S)$. We show that U is open in M. If $W_B=$ $=\hat{\mathscr{J}}(W)$ then $\hat{\mathscr{J}}|W\colon W\to W_B$ is a diffeomorphism and $W_B(y_0)$ is an open neighbourhood of y_0 in ϑ. As in the proof of Theorem 2 there is a bundle map

$$h\colon \bar{\mathscr{J}}^{-1}(W_B(y_0))\to W_B(y_0)\times\mathscr{F}_{y_0}$$

defined by $h(m)=(\bar{\mathscr{J}}(m), g^{-1}(m))$, $m\in\bar{\mathscr{J}}^{-1}(W_B(y_0))$, where $g\in W$ is the unique element for which $\bar{\mathscr{J}}(m)=\hat{\mathscr{J}}(g)(y_0)$ holds. The composite of the mappings

$$W\times\mathscr{F}_{y_0}\xrightarrow{(\hat{\mathscr{J}}|W)\times\mathrm{id}} W_B\times\mathscr{F}_{y_0}\to W_B(y_0)\times\mathscr{F}_{y_0}\xrightarrow{h^{-1}}\bar{\mathscr{J}}^{-1}(W_B(y_0))\subset M$$

is $\pi|W\times\mathscr{F}_{y_0}$ and so $\pi|W\times\mathscr{F}_{y_0}\colon W\times\mathscr{F}_{y_0}\to\bar{\mathscr{J}}^{-1}(W_B(y_0))$ is a diffeomorphism. $W\times S$ is open in $W\times\mathscr{F}_{y_0}$ and thus $U=\pi(W\times S)$ is open in M. By a simple calculation we have $\pi^{-1}(U)=\bigcup_{h\in H_i}(h^{-1}W)\times h(S)$ and because $\pi|W\times S\colon W\times S\to U$ is a diffeomorphism we obtain that $\pi|(h^{-1}W)\times h(S)\colon (h^{-1}W)\times h(S)\to U$ is also a diffeomorphism for every $h\in H_i$. If $h\neq h'$ then $(h^{-1}W)\times h(S)\cap(h'^{-1}W)\times$ $\times h'(S)=\emptyset$ and so U is evenly covered by π. Thus $\pi\colon Q_i\times\mathscr{F}_{y_0}\to M$ is a finite covering. Putting $M_1=Q_i\times\mathscr{F}_{y_0}$ and $\bar{M}_1=\mathscr{F}_{y_0}$ we have

$$\mathbf{T}^q\times\bar{M}_1\approx M_1\xrightarrow{\pi} M$$

where π is a finite covering which accomplishes the proof.

Example. Let G/H be a compact and oriented Riemannian homogeneous space, i.e. G is a compact Lie group, $H\subset G$ is a closed subgroup and the metric tensor of the oriented manifold G/H is induced by a biinvariant metric of G. Then the sectional curvature of G/H is nonnegative for every tangent plane and thus, by [10], Ch. VII. § 23, p. 84, the rank of $\mathscr{J}\colon G/H\to B(G/H)$ is maximal at every point of G/H and $\mathscr{J}$ is surjective. We have $H_B=G_B$ and hence $\hat{\mathscr{J}}\colon Q_i\to G_B$ is a local isomorphism. Because G_i acts transitively on G/H and H_i is a central subgroup, it follows that every isotropy group of the action of Q_i on G/H is trivial. We obtain that Theorem 3 can be applied to $G/H=M$.

3. Applications for Lie groups. In this section every compact Lie group is considered with a biinvariant metric and with the torsionfree connection. Then every compact Lie group belongs to $\mathfrak{P}$.

Theorem 4. *If G is a compact Lie group then $\mathscr{J}$: $G\to B(G)$ is an epimorphism.*

First proof. If $G=T$ is a torus then $B(T)=T$ and $\mathscr{J}_T=\mathrm{id}_T$ hold and the statement is obviously valid. If $G=S$ is a compact semisimple Lie group then its universal covering group $\tilde{S}$ is compact and hence the canonical epimorphism π_S: $\tilde{S}\to S$ is a harmonic map. By § 3

$$\mathscr{J}_S\circ\pi_S = B(\pi_S)\circ\mathscr{J}_{\tilde{S}}$$

holds. The mappings π_S, $\mathscr{J}_S$ and $\mathscr{J}_{\tilde{S}}$ are surjective and so $B(\pi_S)$ is also a surjective map. Thus $0=b_1(\tilde{S})=\dim B(\tilde{S})\geqq\dim B(S)=b_1(S)$ and we obtain that $B(S)$ is a one-point space and $\mathscr{J}_S$ is a constant map. So our statement holds if G is a semisimple Lie group.

In the general case $G=\frac{T\times S}{D}$, where T is a torus S is a semisimple Lie group and $D\subset T\times S$ is a discrete invariant subgroup. By Künneth's formula $b_1(T\times S)=$ $=b_1(T)+b_1(S)$ holds, i.e. $B(T\times S)=B(T)\times B(S)$. It is easy to see that the harmonic map $\mathscr{J}_T\times\mathscr{J}_S$: $T\times S\to B(T)\times B(S)$ is universal among the harmonic maps from $T\times S$ into torus and thus $\mathscr{J}_{T\times S}=\mathscr{J}_T\times\mathscr{J}_S$. Hence $\mathscr{J}_{T\times S}$ is an epimorphism.

Consider the commutative cube in Figure 2, where

$$\pi_D\colon T\times S\to G=\frac{T\times S}{D}$$

denotes the canonical epimorphism. $\mathscr{J}_{T\times S}$ is an epimorphism and so $\tilde{\mathscr{J}}_{T\times S}$ is also an epimorphism. $\tilde{\pi}_D$ is an isomorphism and hence $\tilde{\mathscr{J}}_G=\pi_D^*\circ\tilde{\mathscr{J}}_{T\times S}\circ\tilde{\pi}_D^{-1}$ is an epimorphism. Thus $\mathscr{J}_G$: $G\to B(G)$ is also an epimorphism which accomplishes the proof.

Second proof. Consider the universal covering π_G: $\tilde{G}\to G$ represented by the homotopy classes of curves in G starting from $e\in G$. We have $p_G\circ\tilde{\mathscr{J}}=\mathscr{J}\circ\pi_G$, where p_G: $\mathscr{H}_G^*\to B(G)$ is the canonical projection. $\tilde{\mathscr{J}}$ is a totally geodesic map and hence it commutes with the exponential mapping, i.e. $\tilde{\mathscr{J}}\circ\exp=\exp\circ\tilde{\mathscr{J}}_*$ and $\tilde{\mathscr{J}}(\tilde{e})=0$, where $\tilde{e}\in\tilde{G}$ is the identity element. Denote $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebra of G and $B(G)$, respectively. Because $\mathscr{J}_*$ sends infinitesimal isometries to parallel vector fields, $\tilde{\mathscr{J}}_{*_{\tilde{e}}}$: $\mathfrak{g}\to\mathfrak{h}$ is a Lie algebra homomorphism, where we used the identifications $T_{\tilde{e}}(\tilde{G})=\mathfrak{g}$ and $T_0(\mathscr{H}_G^*)=\mathfrak{h}$. $\tilde{G}$ is simply connected and so, [6], Ch. IV, Sec. 3, Th. 3.1, pp. 54, there exists a Lie group homomorphism $\tilde{h}$: $\tilde{G}\to\mathscr{H}_G^*$ for which $\tilde{h}_{*_{\tilde{e}}}=\tilde{\mathscr{J}}_{*_{\tilde{e}}}$ holds. $\tilde{h}$ commutes with the exponential mapping and hence

$\tilde{\mathscr{J}}=\tilde{h}$. So $\tilde{\mathscr{J}}$ is a homomorphism. By the previous diagram $\mathscr{J}$ is a homomorphism, which accomplishes the proof.

Remark. Denote the kernel of the epimorphism $\mathscr{J}\colon G\to B(G)$ by N. Then N is a compact invariant subgroup of G, $G/N\approx B(G)$ and $\mathscr{J}\colon G\to G/H$ is the quotient map.

Corollary 1. *Let G be a compact Lie group and denote by $P\colon G\times G\to G$ and $A\colon B(G)\times B(G)\to B(G)$ the product and the addition operation respectively. Then*

$$\mathscr{J}_G\circ P = A\circ(\mathscr{J}_G\times\mathscr{J}_G)$$

holds. Moreover, by the identification $B(G\times G)=B(G)\times B(G)$, $B(P)=A$ is valid, where $B(P)$ is defined in § 3.

Let M be compact, oriented and let M', M'' be arbitrary Riemannian manifolds. Further let $g\colon M'\to M''$ be a map. A mapping $f\colon M\to M'$ is said to be g-harmonic if $g_*(\partial f_*)=0$ is valid.

Theorem 5. *Let M be a compact, oriented Riemannian manifold and let G be a compact Lie group. Then every C^2-mapping $f_0\colon M\to G$ is homotopic to a $\mathscr{J}_G$-harmonic mapping $f_1\colon M\to G$. Moreover, there exists a homomorphism $B(f_1)\colon B(M)\to B(G)$ for which $\mathscr{J}_G\circ f_1=B(f_1)\circ\mathscr{J}_M$ holds. If $[M;G]$ denotes the group of homotopy classes from M to G then*

$$B\colon [M;G]\to \mathrm{Hom}\big(B(M),B(G)\big)$$

is a homomorphism.

Proof. By Theorem 4 $\mathscr{J}_G\colon G\to B(G)$ is an epimorphism. Denote the kernel of $\mathscr{J}_G$ by N. Then $G/N\approx B(G)$ holds. Using this identification we may consider $\mathscr{J}_G\colon G\to G/N$ as a principal bundle. Let Γ be a left-invariant connection on this principal bundle. Now let $f_0\colon M\to G$ be an arbitrary C^2-mapping and consider the C^2-mapping $\mathscr{J}_G\circ f_0\colon M\to B(G)$. According to a result of [14] there exists a vector field $v\colon M\to T(B(G))$ along $\mathscr{J}_G\circ f_0$ for which $\exp^{B(G)}\circ v\colon M\to B(G)$ is a harmonic mapping. Now we define a vector field $u\colon M\to T(G)$ along f_0 as follows:

If $m\in M$ then let $u_m\in T_{f_0(m)}(G)$ be a Γ-horizontal vector for which

$$(\mathscr{J}_G)_*u_m = v_m\in T_{\mathscr{J}_G(f_0(m))}\big(B(G)\big)$$

is valid. Furthermore, let $f_t=\exp^G(tu)$, $0\leqq t\leqq 1$.

Because $\mathscr{J}_G\colon G\to B(G)$ is totally geodesic, [4], $(\mathscr{J}_G)_*\circ\partial(f_1)_*=\partial(\mathscr{J}_G\circ f_1)_*$ holds. So, in order to prove that the mapping $f_1\colon M\to G$ is $\mathscr{J}_G$-harmonic it is enough to show that $\exp^{B(G)}\circ v = \mathscr{I}_G\circ f_1$ is valid. If $m\in M$ then

$$\mathscr{I}_G\big(f_1(m)\big) = \mathscr{I}_G\left(\exp^G_{f_0(m)} u_m\right) = \exp^{B(G)}_{\mathscr{I}_G(f_0(m))}\big((\mathscr{I}_G)_{*f_0(m)} u_m\big) = \exp^{B(G)}_{\mathscr{I}_G(f_0(m))} v_m$$

holds, i.e. f_1 is a $\mathscr{I}_G$-harmonic mapping and it is obviously homotopic to f_0.

Because $\mathscr{I}_G\circ f_1\colon M\to B(G)$ is a harmonic mapping there exists an affine mapping

$$B(f_1)\colon B(M)\to B(G)$$

such that $\mathscr{I}_G\circ f_1 = B(f_1)\circ\mathscr{I}_M$ holds. After performing a suitable translation we may suppose that $B(f_1)$ is a homomorphism.

Now let $f_0, h_0\colon M\to G$ be homotopic C^2-mappings and construct $f_1, h_1\colon M\to G$ in the above manner. Then $\mathscr{I}_G\circ f_1$ and $\mathscr{I}_G\circ h_1$ are homotopic harmonic mappings and hence, [14], they can be obtained from each other by suitable translations. So $B(f_1)=B(h_1)$ is valid. Every continuous mapping is homotopic to a C^2-mapping and therefore

$$B\colon [M; G]\to \operatorname{Hom}\big(B(M), B(G)\big)$$

is well-defined. We have to prove that B is a homomorphism. Let $f_0, h_0\colon M\to G$ be C^2-mappings and construct the $\mathscr{I}_G$-harmonic mappings $f_1, h_1\colon M\to G$. We state that $f_1\cdot h_1\colon M\to G$ is also a $\mathscr{I}_G$-harmonic mapping. Indeed, $\mathscr{I}_G\circ(f_1\cdot h_1)=\mathscr{I}_G\circ f_1+$ $+\mathscr{I}_G\circ h_1$ and hence $\mathscr{I}_G\circ(f_1\cdot h_1)$ is a harmonic mapping and $\mathscr{I}_G\circ(f_1\cdot h_1)=$ $=B(f_1\cdot h_1)\circ\mathscr{I}_M$ is valid. So

$$B(f_1\cdot h_1)\circ\mathscr{I}_M = \mathscr{I}_G\circ(f_1\cdot h_1) = \mathscr{I}_G\circ f_1+\mathscr{I}_G\circ h_1 =$$
$$= B(f_1)\circ\mathscr{I}_M + B(h_1)\circ\mathscr{I}_M = \big(B(f_1)+B(h_1)\big)\circ\mathscr{I}_M$$

holds. $B(f_1\cdot h_1)$ is uniquely determined and hence $B(f_1\cdot h_1)=B(f_1)+B(h_1)$ is valid which accomplishes the proof.

The following lemma is a special case of a theorem of H. C. WANG [8], Ch. VI. Sec. 4, Theorem 4.6, p. 248. Here we present a direct proof for this special case.

Lemma. *Let G be a compact Lie group. Then every parallel vector field on G is left-invariant.*

Proof. Let $L_1\subset L_i$ be the ideal of the parallel vector fields of G. If $X\in L_1$ then denote $\tilde X$ the extension of $X_e\in T_e(G)$ to left-invariant vector fields. Then $Y=X-\tilde X\in L_i$ is an infinitesimal isometry and $Y_e=0$. Thus $Y\in I_i$, i.e. Y is tangent to the fibres of the fibration $\mathscr{I}\colon G\to B(G)$. On the other hand the fibres of the epimorphism $\mathscr{I}\colon G\to B(G)$ can be identified with the left-cosets of $N=\ker\mathscr{I}$ and, for every $g\in G$, $(L_g)_*T_e(N)=T_g(gN)$ holds, where L_g denotes the left translation of G with $g\in G$. The vector field $\tilde X$ is orthogonal to gN's, [10]. So Y is orthogonal to the fibres of the fibration $\mathscr{I}\colon G\to B(G)$, i.e. $Y=0$ identically on G. Hence $X=\tilde X$ and so X is left-invariant.

Denote by $\mathfrak{g}$ the Lie algebra of G and let $\mathfrak{z}$ be its center. By the previous lemma $L_1 \subset \mathfrak{g}$. If $X \in L_1$ and $Y \in \mathfrak{g}$ then $[X, Y] = -[Y, X] = -\frac{1}{2}\nabla_Y X = 0$, i.e. $L_1 \subset \mathfrak{z}$ is also valid. Conversely, let $X \in \mathfrak{z}$ and consider an infinitesimal isometry $Y \in L_i$ on G. Then $Y = \sum_{j=1}^{n} \mu_j X^j$, $n = \dim G$ and $X^j \in \mathfrak{g}$, $j = 1, \ldots, n$, and thus $\nabla_Y X = \sum_{j=1}^{n} \mu_j \nabla_{X^j} X = 2 \sum_{j=1}^{n} \mu_j [X^j, X] = 0$, and so $\nabla X = 0$. We obtain that $L_1 = \mathfrak{z}$. Thus $P_i = L_1 = \mathfrak{z}$ can be chosen. If $g \in Q_i$ then there exists $X \in P_i$ with $\exp^{G_i} X = g$. Because $X \in L_1 \subset \mathfrak{g}$ we have $\exp^G X(x) = g(x)$, $x \in G$, and hence the isometry g is identical with a left-translation $L_{\tilde{g}}$ for some $\tilde{g} \in G$. Thus the inclusion $L_1 \subset \mathfrak{g}$ defines an inclusion $Q_i \subset G$ by $g \mapsto \tilde{g}$, $g \in G_i$ and $g = L_{\tilde{g}}$. If $g \in Q_i$ then g commutes with the isometries of G and thus $L_{\tilde{g}}$ commutes with the left translations of G. We obtain that $Q_i \subset Z$ where Z denotes the maximal connected subgroup of the center of G. But $\dim Q_i = \dim P_i = \dim \mathfrak{z} = \dim Z$ and so $Q_i = Z$. By Theorem 4, $\mathscr{I}\colon G \to B(G)$ is an epimorphism. Denote $N \subset G$ the kernel of $\mathscr{I}$. In the proof of Theorem 3 we obtain that $\pi\colon Q_i \times \mathscr{F}_{y_0} \to M$ is a finite covering, where $\pi(g, m) = g(m)$, $g \in Q_i$, $m \in \mathscr{F}_{y_0}$. In our case $G = M$, $e = y_0$, $N = \mathscr{F}_{y_0}$ and Q_i is the center of G. Thus

$$\pi\colon Q_i \times N \to M$$

is an epimorphism with discrete kernel. Because $Q_i \cap N = Q_i(e) \cap \mathscr{F}_{y_0}$ is finite we obtain that N contains finitely many elements of the center Z and hence N is semisimple. In this way we obtain the following classical result:

Theorem 6. *Let G be a compact Lie group. Then $G = \frac{T \times N}{D}$, where T is a toroid, N is a compact semisimple Lie group and D is a finite central subgroup of $T \times N$.*

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